**Spring 2017 MATH5012**

# Exercise 4

(1) Let $\omega_n$ be the volume of the unit ball in $\mathbb{R}^{n+1}$, so $\omega_1 = 2, \omega_2 = \pi, \omega_3 = 4/3\pi$, etc. Show that

$$\omega_n = 2\omega_{n-1} \int_0^1 (1-x^2)^{(n-1)/2} dx,$$

and deduce the formula

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \ .$$

Look up the definition of the Gamma function yourself. This is supposed a problem on Fubini's theorem in advanced calculus.

(2) Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\, dt \quad (x > 0)$$

to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

(3) Complete the following proof of Hardy's inequality (chapter 3, Exercise 14 in [R]): Suppose $f \geq 0$ on $(0, \infty)$, $f \in L^p$, $1 < p < \infty$, and

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt.$$

Write $xF(x) = \displaystyle\int_0^x f(t) t^{\alpha} t^{-\alpha}\, dt$, where $0 < \alpha < 1/q$, use Hölder's inequality to get an upper bound for $F(x)^p$, and integrate to obtain

$$\int_0^\infty F^p(x)\, dx \leq (1-\alpha q)^{1-p} (\alpha p)^{-1} \int_0^\infty f^p(t)\, dt.$$

Show that the best choice of $\alpha$ yields

$$\int_0^\infty F^p(x)\,dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)\,dt.$$

(4) Prove the following analogue of Minkowski's inequality, for $f \ge 0$:

$$\left\{\int \left[f(x,y)\,d\lambda(y)\right]^p \,d\mu(x)\right\}^{\frac{1}{p}} \le \int \left[\int f^p(x,y)\,d\mu(x)\right]^{\frac{1}{p}} d\lambda(y).$$

Supply the required hypotheses.

Many problems are taken from chapter 8, [R].